4. Асимптотические методы

Метод Крылова-Боголюбова. Пример

$$\begin{cases} \ddot{y} - \varepsilon(1 - y^2)\dot{y} + y = 0 & (6) \\ y(0) = y_0, \quad \dot{y}(0) = 0 & (7) \end{cases}$$

$$\begin{cases} \dot{y} = u, \\ \dot{u} = \varepsilon(1 - y^2)u - y, \\ y(0) = y_0, \quad u(0) = 0 \end{cases} \qquad (28)$$

$$\begin{cases} y = a \cdot \cos(t + \theta), \\ u = -a \cdot \sin(t + \theta), \\ a = a(t); \quad \theta = \theta(t). \end{cases} \qquad (29)$$

$$(28),(29) \quad \Rightarrow \quad \dot{a} \cdot \cos(t+\theta) - a \cdot \sin(t+\theta)(1+\dot{\theta}) = -a \cdot \sin(t+\theta),$$

$$-\dot{a} \cdot \sin(t+\theta) - a \cdot \cos(t+\theta)(1+\dot{\theta}) = -\varepsilon(1 - a^2\cos^2(t+\theta)a \cdot \sin(t+\theta)) - a \cdot \cos(t+\theta)$$

$$\dot{a} \cdot \cos(t+\theta) = a \cdot \sin(t+\theta)\dot{\theta},$$

$$\dot{a} \cdot \sin(t+\theta) = -a \cdot \cos(t+\theta)\dot{\theta} + \varepsilon(a \cdot \sin(t+\theta) - a^3 \sin(t+\theta)\cos^2(t+\theta)),$$

$$\dot{a} = \varepsilon(a \cdot \sin^2(t+\theta) - a^3 \sin^2(t+\theta)\cos^2(t+\theta)) =$$

$$= \varepsilon\left(a\frac{1 - \cos 2(t+\theta)}{2} - \frac{a^3}{4}\sin^2 2(t+\theta) \right) =$$

$$= \varepsilon\left(\frac{a}{2} - \frac{a}{2}\cos 2(t+\theta) - \frac{a^3}{4}\frac{1-\cos 4(t+\theta)}{2}\right) =$$

$$= \varepsilon\left(\frac{a}{2} - \frac{a}{2}\cos 2(t+\theta) - \frac{a^3}{8} + \frac{a^3}{8}\cos 4(t+\theta)\right) =$$

$$= \varepsilon\left(\frac{a}{2}\left(1-\frac{a^2}{4}\right) - \frac{a}{2}\cos 2(t+\theta) + \frac{a^3}{8}\cos 4(t+\theta)\right);$$

$$\dot{\theta} = \frac{\dot{a}\cdot\cos(t+\theta)}{a\cdot\sin(t+\theta)} =$$

$$= \frac{\varepsilon(a\cdot\sin^2(t+\theta) - a^3\sin^2(t+\theta)\cos^2(t+\theta))\cos(t+\theta)}{a\cdot\sin(t+\theta)} =$$

$$= \varepsilon\left(\frac{1}{2}\sin 2(t+\theta) - \frac{a^2}{2}\sin 2(t+\theta)\cdot\frac{1+\cos 2(t+\theta)}{2}\right) =$$

$$= \varepsilon\left(\frac{1}{2}\sin 2(t+\theta) - \frac{a^2}{4}\sin 2(t+\theta) - \frac{a^2}{8}\sin 4(t+\theta)\right).$$

$$\begin{cases} \dot{a} = \varepsilon\left\{\frac{a}{2}\left(1-\frac{a^2}{4}\right) - \frac{a}{2}\cos 2(t+\theta) + \frac{a^3}{8}\cos 4(t+\theta)\right\}, \\ \dot{\theta} = \varepsilon\left\{\frac{1}{2}\left(1-\frac{a^2}{2}\right)\sin 2(t+\theta) - \frac{a^2}{8}\sin 4(t+\theta)\right\}, \\ a(0) = y_0, \\ \theta(0) = 0. \end{cases}$$

$$x = (a,\theta)^T; \quad X(x,t) = X_1(x,t) \quad \Rightarrow \quad (30) \Leftrightarrow (9)$$

$$\dot{x} = \varepsilon \cdot X(x,t)$$

$$\xi = (\bar{a},\bar{\theta})^T$$

Первое приближение: $\dot{\xi} = \varepsilon A_1(\xi)$

$$A_1(\varepsilon) = \bar{X}_1(\varepsilon) = \left(\frac{\bar{a}}{2}\left(1-\frac{\bar{a}^2}{4}\right), 0\right)^T;$$

$$\dot{\xi} = \varepsilon A_1(\xi) \Rightarrow (\dot{\bar{a}},\dot{\bar{\theta}})^T = \varepsilon\left(\frac{\bar{a}}{2}\left(1-\frac{\bar{a}^2}{4}\right), 0\right)^T \Rightarrow$$

$$\begin{cases} \dot{\bar{a}} = \varepsilon\dfrac{\bar{a}}{2}\left(1-\dfrac{\bar{a}^2}{4}\right), \\ \bar{a}(0) = y_0, \\ \dot{\bar{\theta}} = 0, \\ \bar{\theta}(0) = 0. \end{cases}$$

(31)

(32)

$$\bar{a} = \frac{2y_0}{\sqrt{y_0^2 + (4 - y_0^2)e^{-\varepsilon t}}}, \quad \bar{\theta} = 0.$$

$$x_1 = \xi \quad \Rightarrow \quad a = \bar{a}; \quad \theta = \bar{\theta} \quad \Rightarrow$$

$$y = \frac{2y_0}{\sqrt{y_0^2 + (4 - y_0^2)e^{-\varepsilon t}}}\cos t$$

$$t \to \infty \quad \Rightarrow \quad y \to 2\cos t$$

$$(21)\quad\Rightarrow\quad \frac{\partial u_1}{\partial t} = X_1(\xi,t) - \bar{X}_1(\xi)\quad\Rightarrow$$

$$(30)\quad\Rightarrow$$

$$X_1(\xi,t) = \begin{pmatrix} \frac{\bar{a}}{2}\left(1-\frac{\bar{a}^2}{4}\right) - \frac{\bar{a}}{2}\cos 2(t+\bar{\theta}) + \frac{\bar{a}^3}{8}\cos 4(t+\bar{\theta}) \\ \frac{1}{2}\left(1-\frac{\bar{a}^2}{2}\right)\sin 2(t+\bar{\theta}) - \frac{\bar{a}^2}{8}\sin 4(t+\bar{\theta}) \end{pmatrix}$$

$$A_1(\xi) = \bar{X}_1(\xi) = \begin{pmatrix} \frac{\bar{a}^2}{2}\left(1-\frac{\bar{a}^2}{4}\right) \\ 0 \end{pmatrix}\quad\Rightarrow$$

$$\frac{\partial u_1}{\partial t} = \begin{pmatrix} -\frac{\bar{a}^2}{2}\cos 2(t+\bar{\theta}) + \frac{\bar{a}^3}{8}\cos 4(t+\bar{\theta}) \\ \frac{1}{2}\left(1-\frac{\bar{a}^2}{2}\right)\sin 2(t+\bar{\theta}) - \frac{\bar{a}^2}{8}\sin 4(t+\bar{\theta}) \end{pmatrix}\qquad(34)$$

$$u_1 = \begin{pmatrix} \frac{\bar{a}^3}{32}\sin 4(t+\bar{\theta}) - \frac{\bar{a}}{4}\sin 2(t+\bar{\theta}) \\ \frac{\bar{a}^2}{32}\cos 4(t+\bar{\theta}) - \frac{1}{4}\left(1-\frac{\bar{a}^2}{2}\right)\cos 2(t+\bar{\theta}) \end{pmatrix}$$

$$a = \bar{a} + \varepsilon\left(\frac{\bar{a}^3}{32}\sin 4(t+\bar{\theta}) - \frac{\bar{a}}{4}\sin 2(t+\bar{\theta})\right)$$

(37)

$$\theta = \bar{\theta} + \varepsilon\left(\frac{\bar{a}^2}{32}\cos 4(t+\bar{\theta}) - \frac{1}{4}\left(1-\frac{\bar{a}^2}{2}\right)\cos 2(t+\bar{\theta})\right)$$

Второе приближение:

$$x_2 = \xi + \varepsilon u_1(\xi,t) + \varepsilon^2 u_2(\xi,t)\qquad(22)$$

$$\dot{\xi} = \varepsilon A_1(\xi) + \varepsilon^2 A_2(\xi)\qquad(23)$$

$$F(\xi,t) = \underset{=0}{X_2(\xi,t)} + \frac{\partial X_1}{\partial \xi} u_1(\xi,t) u_1(\xi,t) - A_1(\xi) \frac{\partial u_1}{\partial \xi}(\xi,t) \qquad (25)$$

$$A_2(\xi) = \overline{F}(\xi,t) \qquad (26)$$

Поскольку выкладки становятся весьма громоздкими, ограничимся тем, что приведем получающиеся в результате уравнения:

$$\dot{\overline{a}} = \varepsilon \frac{\overline{a}}{2}\left(1 - \frac{\overline{a}^2}{4}\right)$$

$$\dot{\overline{\theta}} = -\varepsilon^2 \left(\frac{1}{8} - \frac{\overline{a}^2}{8} + \frac{7\overline{a}^4}{256}\right) \qquad (36)$$

Стационарное решение:

$$\overline{a} = \frac{2y_0}{\sqrt{y_0^2 + (4 - y_0^2)e^{-\varepsilon t}}} \underset{t\to\infty}{\longrightarrow} 2,$$

$$\dot{\overline{\theta}} \underset{t\to\infty}{\longrightarrow} -\frac{\varepsilon^2}{16} \quad \Rightarrow \quad \overline{\theta} \underset{t\to\infty}{\longrightarrow} -\frac{\varepsilon^2}{16} + \theta_0$$