Глава VI. Методы исследования математических моделей

§4. Метод малого параметра. Сингулярные возмущения

23 22 Так как в (17) μ=0

$$(25) \;\Rightarrow\; \mathrm{F}_0(\mathrm{t}) = \mathrm{F}\Big|_{\mu=0} = \mathrm{f}\,(\mathrm{y}_0(\mathrm{t}),\mathrm{t}) = 0 \qquad (26)$$

23 22 26 =0

$$\mathfrak{F}_0(\tau) = \mathfrak{F}\Big|_{\mu=0} = f(y_0(0)+\Pi_0(\tau),0) - f(y_0(0),0) = f(y_0(0)+\Pi_0(\tau),0) \qquad (30)$$

$$(21) \;\Rightarrow\; y(0,\mu) = y_0(0) + \mu y_1(0) + \ldots + \Pi_0(0) + \mu\Pi_1(0) + \ldots =$$

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$$= y^0 = y_0^0 + \mu y_1^0 + \ldots \qquad (31)$$

$$(31) \;\Rightarrow\; \Pi_0(0) = y_0^0 - y_0(0) \qquad (32)$$

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$$\begin{array}{l}(28) \;\Rightarrow \\ (32) \;\Rightarrow\end{array}\begin{cases}\dfrac{d\Pi_0}{d\tau} = \mathfrak{F}_0(\tau) = f(y_0(0)+\Pi_0(\tau),0), \tau > 0, & (33)\\ \Pi_0(0) = y_0^0 - y_0(0) & (34)\end{cases}$$

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$$(27) \;\Rightarrow\; \frac{dy_0}{dt} = F_1(t) = \frac{\partial F(t)}{\partial \mu}\Big|_{\mu=0} = \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \mu}\Big|_{\mu=0} = f_y(y_0(t),t)\,y_1(t) \qquad (35)$$

$$(29) \;\Rightarrow\; \frac{d\Pi_1}{d\tau} = \mathfrak{F}_1(\tau) = \frac{\partial \mathfrak{F}(t)}{\partial\mu}\Big|_{\mu=0} = f_y(y_0(0)+\Pi_0(\tau),0)\frac{\partial y}{\partial \mu}\Big|_{\mu=0} - f_y(y_0(0),0)\frac{\partial y}{\partial\mu}\Big|_{\mu=0} +$$

$$f_t(y_0(0)+\Pi_0(\tau),0)\frac{\partial t}{\partial\mu}\Big|_{\mu=0} - f_t(y_0(0),0)\frac{\partial t}{\partial \mu}\Big|_{\mu=0} =$$

$$= f_y(y_0(0)+\Pi_0(\tau),0)\left(\frac{\partial y_0(\mu\tau)}{\partial t}\underbrace{\frac{\partial t}{\partial \mu}}_{=\tau} + y_1(\mu)\right) + \mu\frac{\partial y_1(\mu\tau)}{\partial t}\underbrace{\frac{\partial t}{\partial\mu}}_{=\tau} + \ldots +$$

$$+\underline{\Pi_1(\tau)} + 2\mu\Pi_2(\tau)\Big|_{\mu=0} - f_y(y_0(0),0)\left(\frac{\partial y_0(\mu\tau)}{\partial t}\underbrace{\frac{\partial t}{\partial\mu}}_{=\tau} + y_1(\mu) + \mu\frac{\partial y_1(\mu\tau)}{\partial t}\underbrace{\frac{\partial t}{\partial \mu}}_{=\tau}\right)\Big|_{\mu=0} +$$

$$+ f_t(y_0(0)+\Pi_0(\tau),0)\underbrace{\frac{\partial t}{\partial \mu}}_{=\tau}\Big|_{\mu=0} - f_t(y_0(0),0)\underbrace{\frac{\partial t}{\partial\mu}}_{=\tau}\Big|_{\mu=0} = f_y(y_0(0)+\Pi_0(\tau),0)\Pi_1(\tau) +$$

$$\boxed{+(f_y(y_0(0)+\Pi_0(\tau),0) - f_y(y_0(0),0)(y_0'(0)\tau + y_1(0)) + (f_t(y_0(0)+\Pi_0(\tau),0) - f_t(y_0(0),0))\tau} \overset{=Q_1}{=}$$

$$= f_y(y_0(0) + \Pi_0(\tau), 0)\Pi_1(\tau) + Q_1 \qquad (36)$$

$$(31) \Rightarrow \; y(0,\mu) = y_0(0) + \underline{\underline{\mu y_1(0)}} + \ldots + \Pi_0(0) + \mu\Pi_1(0) + \ldots =$$

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$$= y^0 = y_0^0 + \underline{\underline{\mu y_1^0}} + \ldots \quad \Rightarrow$$

$$\Pi_1(0) = y_1^{\,0} - y_1(0) \qquad (37)$$

Цепочка решения:

Алгебраическое уравнение:

$$F_0(t) = f(y_0(t), t) = 0 \qquad (26)$$

$$y_0(t)$$

Задача Коши:

$$\begin{cases} \dfrac{d\varPi_0}{d\tau} = f(y_0(0) + \varPi_0(\tau), 0), \tau > 0, & (33) \\ \varPi_0(0) = y_0^0 - y_0(0) & (34) \end{cases}$$

$$\varPi_0(\tau)$$

Алгебраическое уравнение:

$$\frac{dy_0}{dt} = f_y(y_0(t), t) y_1(t) \qquad (35)$$

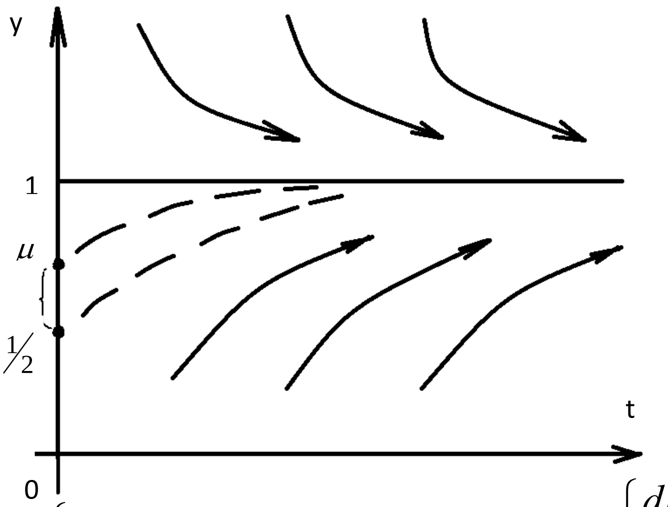
$$y_1(t)$$

Задача Коши:

$$\begin{cases} \dfrac{d\varPi_1}{d\tau} = f_y(y_0(0) + \varPi_0(\tau), 0)\varPi_1(\tau) + Q_1, \tau > 0, & (36) \\ \varPi_1(0) = y_1^0 - y_1(0) \end{cases}$$

$$\varPi_1(\tau)$$

Пример:



$$\begin{cases} \mu \dfrac{dy}{d\tau} = y - y^2, 0 < t \le 1, & (40) \\ y(0) = \dfrac{1}{2} + \mu = y_0^0 + \mu y_1^0 & (41) \end{cases}$$

$(26) \Rightarrow f(y_0(t), t) = 0 \Rightarrow$

$$y_0(t) - y_0^2(t) = 0 \Rightarrow y_0(t) = 1$$

$$\begin{cases} \dfrac{d\Pi_0}{d\tau} = f(y_0(0) + \Pi_0(\tau), 0) \\ \Pi_0(0) = y_0^0 - \underbrace{y_0(0)}_{=1} \end{cases} \Rightarrow \begin{cases} \dfrac{d\Pi_0}{d\tau} = (1 + \Pi_0) - (1 + \Pi_0)^2, \tau > 0, (42) \\ \Pi_0(0) = \dfrac{1}{2} - 1 = -\dfrac{1}{2} \end{cases}$$

$$\boxed{\frac{dy_0}{dt}}^{=0} = f_y(y_0(t), t) y_1(t) \quad (35) \quad \Leftarrow \quad \Pi_0(\tau) = -\frac{1}{1 + e^{\tau}}$$

$$y_1(t) = 0 \Rightarrow y(t, \mu) = \overset{=1}{y_0(t)} + \mu \overset{=0}{y_1(t)} + \Pi_0(\tau) + \mu \Pi_1(\tau) + \ldots =$$

$$= 1 + \Pi_0(\tau) + \underline{\underline{O}}(\mu) = 1 - \frac{1}{1 + e^{\tau}} + \underline{\underline{O}}(\mu) \qquad (44)$$

$$\begin{matrix} (36) \\ (37) \\ (44) \end{matrix} \Rightarrow \begin{cases} \dfrac{d\Pi_1}{d\tau} = \left(1 - 2\left(1 - \dfrac{1}{1 + e^{\tau}}\right)\right)\Pi_1, \tau > 0, \\ f_y = 1 - 2y \\ \Pi_1(0) = y_1^0 - y_1(0) = 1 - 0 = 1 \quad \Rightarrow \end{cases}$$

$$\Pi_1(\tau) = \frac{4e^{\tau}}{(1 + e^{\tau})^2} \Rightarrow y(t, \mu) = \overset{=1}{y_0(t)} + \mu \overset{=0}{y_1(t)} + \Pi_0(\tau) +$$

$$+\mu \Pi_1(\tau) + \underline{\underline{O}}(\mu^2) = 1 - \frac{1}{1+e^{\tau}} + \mu \frac{4e^{\tau}}{(1+e^{\tau})^2} + \underline{\underline{O}}(\mu^2) \quad (48)$$