§2. Общая задача Коши. Функция Римана

Задача Коши для уравнения колебаний с постоянными коэффициентами

$$u_{tt} - u_{zz} + au_t + bu_z + gu = 0$$

$$u = Ue^{-\frac{a}{2}t+\frac{b}{2}z}$$

$$u_t = U_t e^{-\frac{a}{2}t+\frac{b}{2}z} - \frac{a}{2}Ue^{-\frac{a}{2}t+\frac{b}{2}z};\; u_z = U_z e^{-\frac{a}{2}t+\frac{b}{2}z} + \frac{b}{2}Ue^{-\frac{a}{2}t+\frac{b}{2}z}$$

$$u_{tt} = U_{tt}e^{-\frac{a}{2}t+\frac{b}{2}z} - aU_t e^{-\frac{a}{2}t+\frac{b}{2}z} + \frac{a^2}{4}Ue^{-\frac{a}{2}t+\frac{b}{2}z}$$

$$u_{zz} = U_{zz}e^{-\frac{a}{2}t+\frac{b}{2}z} + bU_z e^{-\frac{a}{2}t+\frac{b}{2}z} + \frac{b^2}{4}Ue^{-\frac{a}{2}t+\frac{b}{2}z}$$

$$U_{tt}e^{-\frac{a}{2}t+\frac{b}{2}z} - U_{zz}e^{-\frac{a}{2}t+\frac{b}{2}z} - \cancel{aU_t e^{-\frac{a}{2}t+\frac{b}{2}z}} + \cancel{aU_t e^{-\frac{a}{2}t+\frac{b}{2}z}} + \frac{a^2}{4}Ue^{-\frac{a}{2}t+\frac{b}{2}z} - \frac{a^2}{2}Ue^{-\frac{a}{2}t+\frac{b}{2}z} -$$

$$- \cancel{bU_z e^{-\frac{a}{2}t+\frac{b}{2}z}} + \cancel{bU_z e^{-\frac{a}{2}t+\frac{b}{2}z}} - \frac{b^2}{4}Ue^{-\frac{a}{2}t+\frac{b}{2}z} + \frac{b^2}{2}Ue^{-\frac{a}{2}t+\frac{b}{2}z} + gUe^{-\frac{a}{2}t+\frac{b}{2}z} =$$

$$= \left\{ U_{tt} - U_{zz} - \frac{a^2}{4}U + \frac{b^2}{4}U + gU \right\} e^{-\frac{a}{2}t+\frac{b}{2}z} = 0$$

$$U_{tt} - U_{zz} + \left(g - \frac{a^2}{4} + \frac{b^2}{4} \right) U = 0$$

$$U_{tt} - U_{zz} + CU = 0; \quad C = g - \frac{a^2}{4} + \frac{b^2}{4}$$

$$u\Big|_{t=0} = U\Big|_{t=0} e^{\frac{b}{2}z} = \varphi(z) \Rightarrow U\Big|_{t=0} = \varphi(z)e^{-\frac{b}{2}z} = \varphi_1(z)$$

$$u_t = U_t e^{-\frac{a}{2}t+\frac{b}{2}z} - \frac{a}{2}Ue^{-\frac{a}{2}t+\frac{b}{2}z}$$

$$u_t\Big|_{t=0} = U_t\Big|_{t=0} e^{\frac{b}{2}z} - \frac{a}{2}U\Big|_{t=0} e^{\frac{b}{2}z} = U_t\Big|_{t=0} e^{\frac{b}{2}z} - \frac{a}{2}\varphi(z)e^{-\frac{b}{2}z}e^{\frac{b}{2}z} = \psi(z) \Rightarrow$$

$$U_t\Big|_{t=0} e^{\frac{b}{2}z} = \psi(z) + \frac{a}{2}\varphi(z) \Rightarrow U_t\Big|_{t=0} = (\psi(z) + \frac{a}{2}\varphi(z))e^{-\frac{b}{2}z} = \psi_1(z)$$

Перейдем к переменным x и y:

$$x = t + z;\ y = t - z \Rightarrow t = \frac{x+y}{2},\ z = \frac{x-y}{2}$$

$$U(t,z) = U\left(\frac{x+y}{2}, \frac{x-y}{2}\right) = W(x,y) \Rightarrow$$

$$U_t = W_x + W_y;\ U_{tt} = W_{xx} + 2W_{xy} + W_{yy};\ U_z = W_x - W_y;\ U_{zz} = W_{xx} - 2W_{xy} + W_{yy}$$

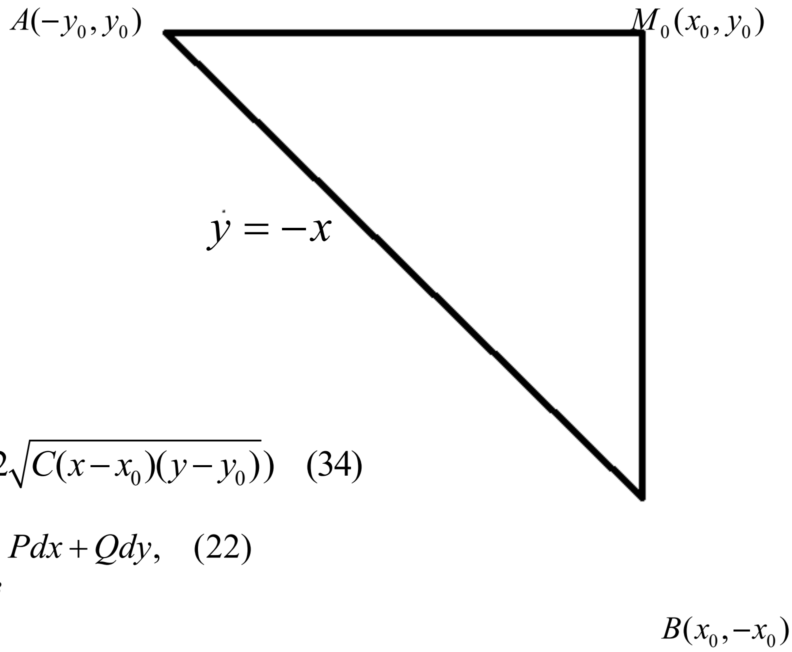
$$U_{tt} - U_{zz} + CU = 0 \ \Rightarrow\ 4W_{xy} + CW = 0 \ \Rightarrow\ W_{xy} + \frac{C}{4}W = 0$$

$$U\Big|_{t=0} = W\Big|_{x+y=0} = \varphi_1\left(\frac{x-y}{2}\right), \quad U_t\Big|_{t=0} = (W_x + W_y)\Big|_{x+y=0} = \psi_1\left(\frac{x-y}{2}\right)$$

$$t = 0 \Rightarrow y = -x;$$

$$x = t + z\Big|_{t=0} = z \Rightarrow dx = dz;$$

$$y = t - z\Big|_{t=0} = -z \Rightarrow dy = -dz;$$



Функция Римана:

$$V(M, M_0) = V(x, y, x_0, y_0) = J_0\left(2\sqrt{C(x - x_0)(y - y_0)}\right) \quad (34)$$

$$W(M_0) = \frac{(\varphi_1 V)_A + (\varphi_1 V)_B}{2} - \frac{1}{2}\int_{AB} Pdx + Qdy, \quad (22)$$

$$\textit{где}\ \ P[WV] = WV_x - W_x V; \quad Q[WV] = W_y V - WV_y,$$

$$f(x,y) = 0 \ \Rightarrow\ W(x_0, y_0) = \frac{\varphi_1(-y_0) + \varphi_1(x_0)}{2} - \frac{1}{2}\int_{AB} (WV_x - W_x V)dx + (W_y V - WV_y)dy \quad (40)$$

$$dy = -dx \ \Rightarrow\ W(x_0, y_0) = \frac{\varphi_1(-y_0) + \varphi_1(x_0)}{2} + \frac{1}{2}\int_{AB} (W_x + W_y)Vdx - \frac{1}{2}\int_{AB} W(V_x + V_y)dx$$

Замечание. На AB аргумент $\dfrac{x-y}{2} = \dfrac{x-(-x)}{2} = x = z$ (*так как* $t = 0$)

$$\frac{\varphi_1(-y_0) + \varphi_1(x_0)}{2} = \frac{\varphi_1(z_0 - t_0) + \varphi_1(z_0 + t_0)}{2}$$

$$\textit{На}\ AB: x + y = 0 \ \Rightarrow\ y = -x \ \Rightarrow\ t = 0 \ \Rightarrow\ x = z;\ dx = dz$$

$$\frac{x-y}{2}=z;\ -y_0=z_0-t_0;\ x_0=z_0+t_0;$$

$$(x-x_0)(y-y_0)=(t+z-t_0-z_0)(t-z-t_0+z_0)=$$
$$=((t-t_0)+(z-z_0))((t-t_0)-(z-z_0))=(t-t_0)^2-(z-z_0)^2 \Rightarrow$$

$$\int_{AB}\psi_1\left(\frac{x-y}{2}\right)V\left(\frac{x-y}{2}\right)dx=\int_{-y_0}^{x_0}\psi_1(x)V(x)dx=\int_{z_0-t_0}^{z_0+t_0}\psi_1(z)J_0(\sqrt{C}\sqrt{t_0^2-(z-z_0)^2})dz$$

$$\int_{AB}W(V_x+V_y)dx=\int_{-y_0}^{x_0}\varphi_1\left(\frac{x-y}{2}\right)(V_x+V_y)dx$$ (y=-x)

$$V_x+V_y\Big|_{y=-x}=\frac{\partial}{\partial t}J_0(\sqrt{C((t-t_0)^2-(z-z_0)^2)})\Big|_{t=0}=$$

$$\frac{J_0'(\sqrt{C(t_0{}^2-(z-z_0)^2)})\sqrt{C}\,2(t-t_0)\big|_{t=0}}{2\sqrt{t_0{}^2-(z-z_0)^2}}=\frac{J_1(\sqrt{C(t_0{}^2-(z-z_0)^2)})\sqrt{C}t_0}{\sqrt{t_0{}^2-(z-z_0)^2}}$$

$$\int_{AB}\varphi_1\left(\frac{x-y}{2}\right)(V_x+V_y)dx=\int_{z_0-t_0}^{z_0+t_0}\varphi_1(z)\frac{J_1(\sqrt{C(t_0{}^2-(z-z_0)^2)})\sqrt{C}t_0}{\sqrt{t_0{}^2-(z-z_0)^2}}dz$$

$$U(z_0,t_0)=\frac{\varphi_1(z_0+t_0)+\varphi_1(z_0-t_0)}{2}+\frac{1}{2}\int_{z_0-t_0}^{z_0+t_0}\left\{\psi_1(z)J_0\left(\sqrt{c}\sqrt{t_0^2-(z-z_0)^2}\right)-\right.$$

$$\left.-\varphi_1(z)\frac{J_1\left(\sqrt{c}\sqrt{t_0^2-(z-z_0)^2}\right)}{\sqrt{t_0^2-(z-z_0)^2}}\sqrt{c}\,t_0\right\}dz.$$

При C=0 из последней формулы получается формула Даламбера (при *a*=1):

$$U\left(z_0,t_0\right)=\frac{\varphi_1\left(z_0+t_0\right)+\varphi_1\left(z_0-t_0\right)}{2}+\frac{1}{2}\int\limits_{z_0-t_0}^{z_0+t_0}\psi_1\left(z\right)dz.$$